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Indecomposable representations of the Poincaré algebra

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Abstract. In this paper we consider indecomposable representations of the Poincaré algebra iso(3, 1) on the space $\Omega = \Omega_+\Omega_-H$ of its universal enveloping algebra. A master representation is obtained on Ω which induces representations on K, the invariant subalgebra of translations, and on Ω_- and Ω_+. These representations are discussed, in particular in view of finite dimensional indecomposable representations of iso(3, 1). The approach taken is analogous to the approach which was chosen by the authors in their analysis of indecomposable representations of the Lorentz algebra so(3, 1). Thus, under restriction of iso(3, 1) to so(3, 1) the earlier results are recovered. The interpretation of the finite dimensional indecomposable representations of iso(3, 1) then follows easily as a coupling of a finite number of irreducible so(3, 1) representations to an indecomposable iso(3, 1) representation, with the dimension of the irreducible representations strictly increasing or strictly decreasing. The bases for the finite dimensional indecomposable iso(3, 1) representations are explicitly determined, and thus also their matrix elements via the inducing representations. A formula for their dimensionalities is obtained.

The methods employed are purely algebraic and follow the line of work of Jacobson and Dixmier.

1. Introduction

In this paper we study certain types of indecomposable representations of the Poincaré algebra iso(3, 1). This work is an extension of our study of indecomposable, as well as irreducible, representations of the Lorentz algebra so(3, 1). Thus, the results obtained in [1] for the Lorentz algebra so(3, 1) will be basic for the study which we carry out in this paper. We will follow [1] closely, and in particular use the same notation as far as possible.

In § 2 of this paper we define the master representation of the Poincaré algebra iso(3, 1) on the space of its universal enveloping algebra Ω, with Ω in a 'natural basis' (i.e. the basis elements of Ω are taken to be tensor products of the basis elements of the algebra iso(3, 1)). The master representation is then reduced to a representation on the space Ω_-, still maintaining the natural basis for Ω_-. Here Ω_- denotes the subspace of Ω which is spanned by the tensor products of the 'lowering operators' of iso(3, 1) alone. Finally, we give a representation on a subspace Ω_k of Ω, where Ω_k is the enveloping algebra of the translation subalgebra of iso(3, 1). In particular, we will obtain formulae for the dimensions of finite dimensional (non-trivial) indecomposable representations induced on quotient spaces of Ω_k with respect to invariant subspaces. The lowest dimensions are 5, 14, 15, ...

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In § 3 we reconsider the representation defined on the space Ω_-, and perform a change of basis in Ω_- from the natural basis to an angular momentum basis. This angular momentum basis is the basis which is most commonly used in physical applications and correspond to the basis used by Gel'fand *et al* [2], Gel'fand and Ponomarev [3], as well as in part of our analysis of so(3, 1) [1].

In § 4 we discuss certain types of indecomposable representations of iso(3, 1), making use of the results of § 3. We will follow closely our discussion of cases A and B as given in [1]. We will obtain, in explicit form, the bases for the finite dimensional indecomposable representations which are induced on quotient spaces of Ω_- with respect to invariant subspaces. Again, we will obtain formulae for the dimensions of these representations. The lowest dimensions obtained here are 5, 8, 11, 13, 14, 14, 17, 18, 20, 20, ...

In § 5 a similar discussion is presented for the case of Ω_+^e. The lowest dimensions of (non-trivial) finite dimensional indecomposable representations obtained on the extended space Ω_+^e (for negative integers n) are 7, 10, 13, 16, 17, 19, ... All the finite dimensional representations obtained in §§ 4 and 5 are in fact representations on the extended space Ω_+^e (or Ω_-^e) with N, n integers.

Our work employs purely algebraic methods, as developed by Jacobson [4], Dixmier [5] and Humphreys [6]. Previous work on indecomposable representations, involving different methods, has been carried out by Angelopoulos [7], Paneitz [8], Raczka [9] and Bertrand and Rideau [10].

2. Master representation

We choose for the Poincaré algebra iso(3, 1) the basis

$$\{h_3, h_+, h_-, p_3, p_+, p_-, k_0, k_3, k_+, k_-\}. \tag{2.1}$$

The elements h correspond to the (angular momentum) subalgebra so(3), the elements h and p to the (Lorentz) subalgebra so(3, 1) and the elements k to the invariant (translation) subalgebra K.

The non-vanishing Lie products are given in this basis by

$$\begin{array}{lll}
[h_3, h_\pm] = \pm h_\pm & [h_+, h_-] = 2h_3 & \\
[h_3, p_\pm] = \pm p_\pm & [h_+, p_-] = [p_+, h_-] = 2p_3 & [p_3, h_\pm] = \pm p_\pm \\
[p_3, p_\pm] = \mp h_\pm & [p_+, p_-] = -2h_3 & \\
[k_3, h_\pm] = \pm k_\pm & [h_3, k_\pm] = \pm k_\pm & \\
[k_0, p_\pm] = -k_\pm & [p_3, k_3] = -k_0 & [p_3, k_0] = k_3 \\
[h_\pm, k_\mp] = \pm 2k_3 & [p_\pm, k_\mp] = -2k_0. &
\end{array} \tag{2.2}$$

The above relations are valid only if all upper signs or all lower signs are taken simultaneously.

Another basis which is frequently used in physical applications is

$$\{M_{\mu\nu}, P_\alpha\,; \mu, \nu, \alpha = 0, 1, 2, 3\}$$

with Lie products

$$[P_\mu, P_\nu] = 0 \qquad [M_{\mu\nu}, P_\alpha] = \mathrm{i}(g_{\mu\alpha}P_\nu - g_{\nu\alpha}P_\mu)$$

$$[M_{\mu\nu}, M_{\alpha\beta}] = \mathrm{i}(g_{\mu\alpha}M_{\nu\beta} - g_{\nu\alpha}M_{\mu\beta} + g_{\mu\beta}M_{\alpha\nu} - g_{\nu\beta}M_{\alpha\mu})$$

where

$$g_{00}=-g_{11}=-g_{22}=-g_{33}=-1 \qquad g_{\mu\nu}=0 \text{ if } \mu\neq\nu.$$

This basis is related to ours by

$$\begin{aligned}
&h_+=-M_{23}-\mathrm{i}M_{13} & &\\
&h_-=-M_{23}+\mathrm{i}M_{13} & &k_+=-P_2-\mathrm{i}P_1\\
&h_3=-M_{12} & &k_-=P_2-\mathrm{i}P_1\\
&p_3=M_{03} & &k_3=-\mathrm{i}P_3\\
&p_+=M_{01}-\mathrm{i}M_{02} & &k_0=P_0\\
&p_-=M_{01}+\mathrm{i}M_{02}. & &
\end{aligned}$$

We choose the following 'natural basis' for the universal enveloping algebra Ω of iso(3, 1):

$$\Omega\colon \{X(u,m,p,s,n,q,z,w,t,r)=p_-^u h_-^m k_-^p p_+^s h_+^n k_+^q k_3^z k_0^w p_3^t h_3^r,\ u,m,\ldots\in\mathbb{N}\} \tag{2.3}$$

where the product is the ordered tensor product and $X(0,0,\ldots,0)=\mathbb{1}$ denotes the identity operator. Apart from the Lie products of [1], equation (2.3), taken within the universal enveloping algebra of so(3, 1), one needs the following Lie products, taken within the universal enveloping algebra of iso(3, 1),

$$\begin{aligned}
&[k_3,h_\pm^m]=\pm m h_\pm^{m-1}k_\pm & &[k_0,p_\pm^m]=-mp_\pm^{m-1}k_\pm\\
&[k_\pm,h_\mp^m]=\pm 2mh_\mp^{m-1}k_3-m(m-1)h_\mp^{m-2}k_\mp & &\\
&[k_\pm,p_\mp^m]=2mp_\mp^{m-1}k_0-m(m-1)p_\mp^{m-2}k_\mp & &\\
&[h_3,k_\pm^m]=\pm mk_\pm^m & &[h_\pm,k_3^m]=\mp mk_3^{m-1}k_\pm\\
&[h_\mp,k_\pm^m]=\mp 2mk_\pm^{m-1}k_3 & &\\
&[p_3,k_3^m]=-mk_3^{m-1}k_0 & &[p_3,k_0^m]=mk_0^{m-1}k_3\\
&[p_\mp,k_\pm^m]=-2mk_\pm^{m-1}k_0 & &[p_\pm,k_0^m]=mk_0^{m-1}k_\pm.
\end{aligned} \tag{2.4}$$

Making use of these relations one obtains the master representation ρ of the Poincaré algebra iso(3, 1) on the space of its universal enveloping algebra Ω in the natural basis, equation (2.3):

$$\rho(h_3)X=X(r+1)+(n+s+q-u-m-p)X$$

$$\begin{aligned}\rho(p_3)X=X(t+1)&+wX(w-1,z+1)-zX(z-1,w+1)+nX(s+1,n-1)\\
&-sX(n+1,s-1)-mX(u+1,m-1)+uX(u-1,m+1)\end{aligned}$$

$$\rho(p_-)X=X(u+1)$$

$$\rho(h_-)X=X(m+1)$$

$$\begin{aligned}\rho(p_+)X=X(s+1)&+u(u-1-2q-2n-2s+2p+2m)X(u-1)\\
&+2psX(p-1,s-1,q+1)-2pX(p-1,w+1)\\
&+2mX(m-1,t+1)+2mwX(m-1,w-1,z+1)\\
&-2mzX(m-1,z-1,w+1)+2mnX(m-1,s+1,n-1)\end{aligned}$$

$$
\begin{aligned}
&-2msX(m-1, n+1, s-1)\\
&-m(m-1)X(u+1, m-2)-2uX(u-1, r+1)\\
\rho(h_+)X &= X(n+1)+m(-m+1+2q+2n+2s-2p-2u)X(m-1)\\
&+2pX(p-1, z+1)+2pnX(p-1, n-1, q+1)+2mX(m-1, r+1)\\
&+2uX(u-1, t+1)+2uwX(u-1, w-1, z+1)\\
&-2uzX(u-1, w+1, z-1)+2unX(u-1, n-1, s+1)\\
&-2usX(u-1, s-1, n+1)+u(u-1)X(u-2, m+1)\\
\rho(k_3)X &= X(z+1)+nX(n-1, q+1)-mX(m-1, p+1)\\
\rho(k_0)X &= X(w+1)-sX(s-1, q+1)-uX(u-1, p+1)\\
\rho(k_-)X &= X(p+1)\\
\rho(k_+)X &= X(q+1)+2mX(m-1, z+1)+2mnX(m-1, n-1, q+1)\\
&-m(m-1)X(m-2, p+1)+2uX(u-1, w+1)\\
&-2usX(u-1, s-1, q+1)-u(u-1)X(u-2, p+1).
\end{aligned}
\tag{2.5}
$$

Imposing the conditions

$$
\begin{aligned}
&\rho(h_+)\mathbb{1}=\rho(p_+)\mathbb{1}=\rho(k_+)\mathbb{1}=\rho(k_3)\mathbb{1}=\rho(k_0)\mathbb{1}=0\\
&\rho(h_3)\mathbb{1}=\Lambda_1\mathbb{1} \qquad \rho(p_3)\mathbb{1}=\Lambda_2\mathbb{1}
\end{aligned}
\tag{2.6}
$$

the master representation, equation (2.5), induces on the space Ω_- of the lowering algebra with basis

$$
\Omega_-: \{X(u, m, p)=p_-^u h_-^m k_-^p,\ u, m, p \in \mathbb{N}\}
\tag{2.7}
$$

the representation

$$
\begin{aligned}
\rho(h_3)X &= (\Lambda_1-u-m-p)X\\
\rho(p_3)X &= \Lambda_2 X-mX(u+1, m-1)+uX(u-1, m+1)\\
\rho(p_-)X &= X(u+1)\\
\rho(h_-)X &= X(m+1)\\
\rho(p_+)X &= u(u-1+2p+2m)X(u-1)-m(m-1)X(u+1, m-2)+2m\Lambda_2X(m-1)\\
&-2u\Lambda_1X(u-1)\\
\rho(h_+)X &= m(-m+1-2p-2u)X(m-1)+u(u-1)X(u-2, m+1)+2m\Lambda_1X(m-1)\\
&+2u\Lambda_2X(u-1)\\
\rho(k_3)X &= -mX(m-1, p+1)\\
\rho(k_0)X &= -uX(u-1, p+1)\\
\rho(k_-)X &= X(p+1)\\
\rho(k_+)X &= -m(m-1)X(m-2, p+1)-u(u-1)X(u-2, p+1).
\end{aligned}
\tag{2.8}
$$

It is this representation which will be discussed in detail in the following sections, after a change of basis to an angular momentum basis has been performed.

The translation subalgebra K is an invariant subalgebra. Thus, we obtain a representation on the space Ω_k with basis

$$\Omega_k: \{X(p, q, z, w) = k_-^p k_+^q k_3^z k_0^w,\ p, q, z, w \in \mathbb{N}\}. \tag{2.9}$$

A representation on this space is obtained as ($\rho(h_3)\mathbb{1} = \rho(p_3)\mathbb{1} = 0$, $u = m = s = n = 0$)

$$\begin{aligned}
\rho(h_3)X &= (q-p)X \\
\rho(p_3)X &= wX(z+1, w-1) - zX(z-1, w+1) \\
\rho(h_-)X &= zX(p+1, z-1) - 2qX(q-1, z+1) \\
\rho(p_-)X &= wX(p+1, w-1) - 2qX(q-1, w+1) \\
\rho(h_+)X &= -zX(q+1, z-1) + 2pX(p-1, z+1) \\
\rho(p_+)X &= wX(q+1, w-1) - 2pX(p-1, w+1) \\
\rho(k_3)X &= X(z+1) \\
\rho(k_0)X &= X(w+1) \\
\rho(k_-)X &= X(p+1) \\
\rho(k_+)X &= X(q+1).
\end{aligned} \tag{2.10}$$

This representation is infinite dimensional and indecomposable. It contains an infinity of invariant subspaces which are nested into each other (composition series). It is seen that

$$N = N_1 + N_2 = p + q + z + w \qquad N, p, q, z, w \in \mathbb{N}$$

remains constant under the action of so(3, 1), while

$$N_1 = p + q + z \qquad N_1, p, q, z \in \mathbb{N}$$

remains constant under the action of so(3). Thus, since $N = 0, 1, 2, 3, \ldots$ for iso(3, 1), it follows that this indecomposable iso(3, 1) representation contains irreducible so(3, 1) representations for $N = 0, 1, 2, 3, \ldots$. Each of the irreducible so(3, 1) representations contains in turn irreducible so(3) representations with a multiplicity $\geqslant 1$. For a given value of N the dimension of the so(3, 1) representation is

$$\tfrac{1}{6}(N+1)(N+2)(N+3).$$

The action of the invariant subalgebra K always increases the value of N. Thus, each value of N defines an infinite dimensional invariant subspace V_N of the representation equation (2.10), with $V_0 \equiv \Omega_k$. Thus, this representation of iso(3, 1) induces on the quotient spaces Ω_k / V_{N+1}, $N \geqslant 1$, finite dimensional indecomposable representations. Still other finite dimensional indecomposable representations are obtained by observing that the invariant subspace V_M contains the invariant subspace V_N, $M < N$. On the quotient space V_M / V_{N+1}, $M < N$, one obtains the finite dimensional indecomposable iso(3, 1) representations of dimension

$$\begin{aligned}
&\dim \rho(M, N) = \tfrac{1}{24}[(N+1)(N+2)(N+3)(N+4) - M(M+1)(M+2)(M+3)] \\
&\qquad N = 1, 2, 3, \ldots, \qquad M = 0, 1, 2, \ldots, N-1.
\end{aligned} \tag{2.11}$$

Note that for $M = 0$ one obtains the finite dimensional indecomposable representations on Ω_k / V_{N+1}.

The representations of dimension $\leqslant 20$ are

$$5=(3+1)+1 \qquad 14=(5+1)+(3+1)+4 \qquad 15=(5+1)+(3+1)+4+1$$

where the sums give the so(3, 1) content, with the so(3) content in parentheses.

3. Representations on the 'lowering' algebra Ω_-

In this section we perform a change of basis of the space Ω_- to an angular momentum basis and study the representations of iso(3, 1) on the space Ω_- in this new basis.

We define $\rho(h_+)$ extremal vectors through the condition

$$\begin{aligned} &\rho(h_+)y_{Nq}=0 \\ &y_{Nq}=\sum_{\substack{k\geqslant 0\\ k+q\leqslant N}} c_{kq}X(N-k-q,k,q). \end{aligned} \tag{3.1}$$

One obtains the following recurrence relations for the coefficients c_{kq}:

$$\begin{aligned} c_{1q}&=-\frac{2\Lambda_2(N-q)}{2\Lambda_1+2-2N} \\ c_{kq}&=-\frac{(N-k-q+2)(N-k-q+1)}{k(2\Lambda_1-2N+k+1)}c_{k-2,q}-\frac{2\Lambda_2(N-k-q+1)}{k(2\Lambda_1-2N+k+1)}c_{k-1,q} \end{aligned} \tag{3.2}$$

for each $q\in\mathbb{N}$.

The angular momentum basis for Ω_- is given by

$$\{y^m_{Nq}\equiv h^m_- y_{Nq},\ m\in\mathbb{N},\ q\in\mathbb{N},\ N-q\in\mathbb{N}\}. \tag{3.3}$$

Making use of the results of [1], and again applying the process of induction, the representations of iso(3, 1) on Ω_- in an angular momentum basis are obtained as

$$\begin{aligned} &\rho(h_3)y^m_{Nq}=(\Lambda_1-N-m)y^m_{Nq} \quad \rho(h_+)y^m_{Nq}=m(2\Lambda_1-2N+1-m)y^{m-1}_{Nq} \quad \rho(h_-)y^m_{Nq}=y^{m+1}_{Nq} \\ &\rho(p_3)y^m_{Nq}=\alpha_{Nq}(2\Lambda_1-2N+1-m)y^{m+1}_{N-1,q} \\ &\qquad\qquad +\beta_{Nq}(\Lambda_1-N-m)y^m_{Nq}-my^{m-1}_{N+1,q} \\ &\rho(p_+)y^m_{Nq}=-\alpha_{Nq}(2\Lambda_1-2N+1-m)(2\Lambda_1-2N+2-m)y^m_{N-1,q} \\ &\qquad\qquad +\beta_{Nq}m(2\Lambda_1-2N+1-m)y^{m-1}_{Nq}-m(m-1)y^{m-2}_{N+1,q} \\ &\rho(p_-)y^m_{Nq}=\alpha_{Nq}y^{m+2}_{N-1,q}+\beta_{Nq}y^{m+1}_{Nq}+y^m_{N+1,q} \\ &\rho(k_-)y^m_{Nq}=\delta_{Nq}y^{m+2}_{N-1,q+1}+\gamma_{Nq}y^{m+1}_{N,q+1}+y^m_{N+1,q+1} \qquad \rho(k_0)y^m_{Nq}=-(N-q)y^m_{N,q+1} \\ &\rho(k_3)y^m_{Nq}=(2\Lambda_1-2N+1-m)\delta_{Nq}y^{m+1}_{N-1,q+1}+(\Lambda_1-N-m)\gamma_{Nq}y^m_{N,q+1}-my^{m-1}_{N+1,q+1} \\ &\rho(k_+)y^m_{Nq}=-(2\Lambda_1-2N+1-m)(2\Lambda_1-2N+2-m)\delta_{Nq}y^m_{N-1,q+1} \\ &\qquad\qquad +m(2\Lambda_1-2N+1-m)\gamma_{Nq}y^{m-1}_{N,q+1}-m(m-1)y^{m-2}_{N+1,q+1} \end{aligned} \tag{3.4}$$

where

$$\alpha_{Nq}=\frac{[\Lambda_2^2+(\Lambda_1+1-N)^2](N-q)(2\Lambda_1+2-N-q)}{(\Lambda_1+1-N)^2(2\Lambda_1-2N+3)(2\Lambda_1-2N+1)}$$

$$\beta_{Nq} = \frac{\Lambda_2(\Lambda_1 + 1 - q)}{(\Lambda_1 - N)(\Lambda_1 + 1 - N)}$$

$$\delta_{Nq} = \frac{(N-q)(N-q-1)[\Lambda_2^2 + (\Lambda_1 + 1 - N)^2]}{(2\Lambda_1 - 2N + 1)(2\Lambda_1 - 2N + 3)(\Lambda_1 - N + 1)^2}$$

$$\gamma_{Nq} = \frac{(N-q)\Lambda_2}{(\Lambda_1 - N)(\Lambda_1 - N + 1)}.$$

Although these relations were derived on the space Ω_- with basis equation (3.3), i.e. $N-q$, q, $m \in \mathbb{N}$, they are seen to be valid on the extended space Ω_-^e for N, q, $m \in \mathbb{Z}$ (though then y_{Nq} loses its former meaning and represents some abstract basis).

One observes the following properties of this representation.

(i) The action of the Lorentz subalgebra so(3, 1) does not affect the parameter q. Thus, for fixed value of q the relations (3.3) yield representations of the subalgebra so(3, 1) which are identical to the representations which were obtained in [1]. In fact, for fixed q, the substitution

$$\Lambda_1 - q \to \Lambda_1 \qquad N - q \to N$$

brings the so(3, 1) representations as defined above into the form given in [1]. Thus the set of basis elements

$$\{y^m_{Nq}, q \text{ fixed}\}$$

corresponds to the so(3, 1) basis of [1]

$$\{y^m_N, N \geq q\}.$$

(ii) The action of the translation subalgebra K *always* increases the value of q by 1. This then causes an indecomposability in the parameter q. In fact, the action of $\rho(k_3)$, $\rho(k_+)$, $\rho(k_-)$ is *identical* to the action of the operators $\rho(p_3)$, $\rho(p_+)$, $\rho(p_-)$, respectively, in regard to the indices N, m, while it increases the value of q by 1. The action of $\rho(k_0)$ does not affect the values N, m but merely increases the value of q by 1 (see figure 1). This follows from the fact that k_+, k_-, k_3 are the three components of an $l = 1$ angular momentum operator while k_0 is an $l = 0$ angular momentum operator.

(iii) The coefficients δ, γ differ from coefficients α, β, respectively, only by certain factors, namely

$$\delta_{Nq} = \frac{N-q-1}{2\Lambda_1 + 2 - N - q}\alpha_{Nq} \qquad \gamma_{Nq} = \frac{N-q}{\Lambda_1 + 1 - q}\beta_{Nq}.$$

4. Finite dimensional indecomposable representations

In this section we want to discuss the representation given by equation (3.4) for certain specific values of the parameters Λ_1 and Λ_2. We are primarily interested in the finite dimensional indecomposable representations. These are obtained on certain quotient spaces. To obtain the bases for these representations one has to go through the analysis of infinite dimensional indecomposable representations following closely our previous work in [1]. We refer the reader to [1] and appendices 1 and 2 for a more detailed presentation.

Let us only mention here that finite dimensional indecomposable representations arise due to the 'staircase' invariant subspaces. For a fixed value of q the elements of

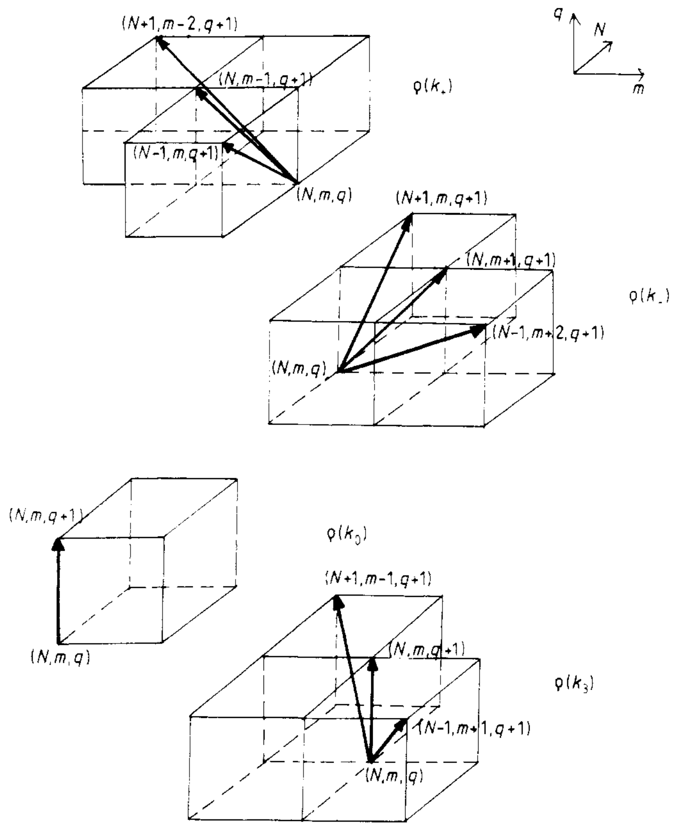

Figure 1. The action of $\rho(k_+)$, $\rho(k_-)$, $\rho(k_0)$, $\rho(k_3)$.

an so(3, 1) invariant subspace are the elements y^m_{Nq} which lie above the 'staircase' defined by the key equation $2\Lambda_1 - 2N + 1 - m = 0$. Finite dimensional indecomposable representations are obtained on the quotient spaces modulo these invariant subspaces.

It may be useful to consider an example at this point. We choose $M = 3$, $n = 1$. The union of the sets of elements: $\{y^{m+4}_{00}, y^{m+2}_{10}, y^m_{2+N,0}, m, N \in \mathbb{N}\}$, $\{y^{2+m}_{11}, y^m_{2+N,1}, m, N \in \mathbb{N}\}$ and V^q, $q = 2, 3, 4, \ldots$, forms a basis for an iso(3, 1) invariant subspace of Ω_- (see appendix 2). The quotient space of Ω_- with respect to this invariant subspace then has a basis $\{y^0_{00}, y^1_{00}, y^2_{00}, y^3_{00}, y^0_{10}, y^1_{10}, y^0_{11}, y^1_{11}\}$. With respect to so(3, 1) this quotient space carries a six-dimensional ($q = 0$) and a two-dimensional ($q = 1$) irreducible representation while for iso(3, 1) this space carries an eight-dimensional indecomposable representation whose invariant subspace is spanned by $\{y^0_{11}, y^1_{11}\}$. In explicit form:

$$\rho(h_3) = \begin{bmatrix} 3/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 \end{bmatrix}$$

$$\rho(h_+)=\begin{bmatrix} 0&3&0&0&0&0&0&0\\ 0&0&4&0&0&0&0&0\\ 0&0&0&3&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$\rho(h_-)=\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0 \end{bmatrix}$$

$$\rho(p_3)=\begin{bmatrix} i/2&0&0&0&0&0&0&0\\ 0&i/6&0&0&8/9&0&0&0\\ 0&0&-i/6&0&0&4/9&0&0\\ 0&0&0&-i/2&0&0&0&0\\ 0&0&0&0&5i/6&0&0&0\\ 0&0&0&0&0&-5i/6&0&0\\ 0&-1&0&0&0&0&i/2&0\\ 0&0&-2&0&0&0&0&-i/2 \end{bmatrix}$$

$$\rho(p_+)=\begin{bmatrix} 0&i&0&0&-8/3&0&0&0\\ 0&0&4i/3&0&0&-8/9&0&0\\ 0&0&0&i&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&-2&0&0&5i/3&0&0\\ 0&0&0&-6&0&0&0&0\\ 0&0&0&0&0&0&0&i\\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$\rho(p_-)=\begin{bmatrix} 0&0&0&0&0&0&0&0\\ i/3&0&0&0&0&0&0&0\\ 0&i/3&0&0&4/9&0&0&0\\ 0&0&i/3&0&0&4/9&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&5i/3&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&i&0 \end{bmatrix}$$

$$\rho(k_-)=\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&2\mathrm{i}/3&0&0&0 \end{bmatrix}$$

$$\rho(k_+)=\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&-2&0&0&2\mathrm{i}/3&0&0\\ 0&0&0&-6&0&0&0&0 \end{bmatrix}$$

$$\rho(k_3)=\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&-1&0&0&\mathrm{i}/3&0&0&0\\ 0&0&-2&0&0&-\mathrm{i}/3&0&0 \end{bmatrix}$$

$$\rho(k_0)=\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0 \end{bmatrix}.$$

In the evaluation of the matrices we used:

$$\alpha_{00}=\alpha_{11}=\delta_{00}=\delta_{11}=\delta_{10}=\gamma_{00}=\gamma_{11}=0, \qquad \beta_{00}=\mathrm{i}/3,$$
$$\beta_{10}=5\mathrm{i}/3, \qquad \beta_{11}=\mathrm{i}, \qquad \alpha_{10}=4/9, \qquad \gamma_{10}=2\mathrm{i}/3.$$

In a similar manner other finite dimensional indecomposable representations can be obtained in an explicit form. In what follows we list the bases, as well as the so(3, 1) content for the representations with small dimensions.

For $\Lambda_1 = M/2$, $\Lambda_2 = \pm \mathrm{i}n/2$ ($M, n = 3, 5, 7, \ldots\, n < M$), case A4 in appendix 1, the bases for finite dimensional quotient spaces are given by

$$T_{(M/2, n/2, q_c)} = \{y^m_{Nq}, q \leqslant N \leqslant (M-n)/2, 0 \leqslant m \leqslant M - 2N, 0 \leqslant q \leqslant q_c\} \tag{4.1}$$

where $0 \leqslant q_c \leqslant (M-n)/2$. The number q_c signifies the number of q levels (so(3, 1) irreducible representations) which occur in the iso(3, 1) representation.

It is thus seen that to each finite dimensional representation there corresponds a triplet of numbers $(M/2, n/2, q_c)$. A general expression for the dimensions of finite dimensional indecomposable representations of iso(3, 1) will be given shortly.

Below we list the dimension and the so(3, 1) content of some of the smallest iso(3, 1) representations $(M/2, n/2, q_c) = (\Lambda_1, -\mathrm{i}\Lambda_2, q_c)$:

(3/2, 1/2, 1)	8 = 6 + 2 = (4 + 2) + 2
(5/2, 3/2, 1)	14 = 10 + 4 = (6 + 4) + 4
(5/2, 1/2, 1)	18 = 12 + 6 = (6 + 4 + 2) + (4 + 2)
(7/2, 5/2, 1)	20 = 14 + 6 = (8 + 6) + 6
(5/2, 1/2, 2)	20 = 12 + 6 + 2 = (6 + 4 + 2) + (4 + 2) + 2
(9/2, 7/2, 1)	26 = 18 + 8 = (10 + 8) + 8
(7/2, 3/2, 1)	28 = 18 + 10 = (8 + 6 + 4) + (6 + 4)
(7/2, 1/2, 1)	32 = 20 + 12 = (8 + 6 + 4 + 2) + (6 + 4 + 2)
(7/2, 3/2, 2)	32 = 18 + 10 + 4 = (8 + 6 + 4) + (6 + 4) + 4
(11/2, 9/2, 1)	32 = 22 + 10 = (12 + 10) + 10

etc.

For $\Lambda_1 = M$, $\Lambda_2 = \pm \mathrm{i}n$ ($M, n = 1, 2, 3, \ldots, n < M$), case B3 in appendix 1, the bases for the finite dimensional quotient spaces are given by

$$T_{(M, n, q_c)} = \{y^m_{Nq}, q \leqslant N \leqslant M - n, 0 \leqslant m \leqslant 2M - 2N, 0 \leqslant q \leqslant q_c\} \tag{4.2}$$

where $0 \leqslant q_c \leqslant M - n$. The value q_c again gives the number of q levels which occur in the iso(3, 1) representation.

The few lowest dimensional representations obtained this way are listed below with $(M, n, q_c) = (\Lambda_1, -\mathrm{i}\Lambda_2, q_c)$:

(1, 0, 1)	5 = 4 + 1 = (3 + 1) + 1
(2, 1, 1)	11 = 8 + 3 = (5 + 3) + 3
(2, 0, 1)	13 = 9 + 4 = (5 + 3 + 1) + (3 + 1)
(2, 0, 2)	14 = 9 + 4 + 1 = (5 + 3 + 1) + (3 + 1) + 1
(3, 2, 1)	17 = 12 + 5 = (7 + 5) + 5
(4, 3, 1)	23 = 16 + 7 = (9 + 7) + 7
(3, 1, 1)	23 = 15 + 8 = (7 + 5 + 3) + (5 + 3)
(3, 0, 1)	25 = 16 + 9 = (7 + 5 + 3 + 1) + (5 + 3 + 1)
(3, 1, 2)	26 = 15 + 8 + 3 = (7 + 5 + 3) + (5 + 3) + 3

$$(3,0,2) \qquad 29=16+9+4=(7+5+3+1)+(5+3+1)+(3+1)$$
$$(5,4,1) \qquad 29=20+9=(11+9)+9$$
$$(3,0,3) \qquad 30=16+9+4+1=(7+5+3+1)+(5+3+1)=1$$

etc.

One may notice that the formulae for the Poincaré algebra representations derived on the basis y^m_{Nq} for m, q, $N \in \mathbb{N}$ and $N \geqslant q$ can be extended to all integral values of m, q and N. Our original space Ω_- then becomes an invariant subspace of a certain abstract space Ω^e_- on which the analysis can be developed. In our geometrical picture Ω^e_- would correspond to $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$.

5. Representations on the 'raising' algebra Ω_+

In order to obtain the representations ρ' of iso(3, 1) in the angular momentum basis we choose as the basis for Ω the ordered set:

$$\Omega\colon \{Y(s,n,q,u,m,p,z,w,t,r) = p^s_+ h^n_+ k^q_+ p^u_- h^m_- k^p_- k^z_3 k^w_0 p^t_3 h^r_3,\ s,n,q,u,m,p,z,w,t,r \in \mathbb{N}\}. \tag{5.1}$$

The basis for Ω_+ then becomes

$$\Omega_+\colon \{Y(s,n,q),\ s,n,q \in \mathbb{N}\}$$

where

$$Y(s,n,q) = p^s_+ h^n_+ k^q_+.$$

The basis X for Ω_- goes over into the basis Y for Ω_+ under the Lie algebra automorphism

$$\begin{array}{lll} h_3 \to -h_3 & h_+ \to h_- & h_- \to h_+ \\ p_3 \to -p_3 & p_+ \to p_- & p_- \to p_+ \\ k_3 \to -k_3 & k_+ \to k_- & k_- \to k_+ \\ k_0 \to k_0. & & \end{array} \tag{5.2}$$

The representations ρ' of iso(3, 1) on Ω and Ω_+ are then obtained from the representations ρ by the substitution

$$(\Lambda_1, \Lambda_2) \to (-\Lambda_1, -\Lambda_2)$$

and

$$\begin{array}{ll} \rho'(h_3) = -\rho(h_3) & \rho'(h_+) = \rho(h_-) \\ \rho'(p_3) = -\rho(p_3) & \rho'(h_-) = \rho(h_+) \\ \rho'(k_3) = -\rho(k_3) & \rho'(k_+) = \rho(k_-) \\ \rho'(k_0) = \rho(k_0) & \rho'(k_-) = \rho(k_+) \\ & \rho'(p_+) = \rho(p_-) \\ & \rho'(p_-) = \rho(p_+). \end{array} \tag{5.3}$$

Representation on Ω_+:

$$\rho'(h_3)y^n_{Np} = (N+n+\Lambda_1)y^n_{Np} \qquad \rho'(h_+)y^n_{Np} = y^{n+1}_{Np}$$
$$\rho'(h_-)y^n_{Np} = n(-2\Lambda_1-2N-n+1)y^{n-1}_{Np}$$
$$\rho'(p_3)y^n_{Np} = -\alpha_{Np}(-2\Lambda_1-2N-n+1)y^{n+1}_{N-1,p}+\beta_{Np}(\Lambda_1+N+n)y^n_{Np}+ny^{n-1}_{N+1,p}$$
$$\rho'(p_+)y^n_{Np} = \alpha_{Np}y^{n+2}_{N-1,p}+\beta_{Np}y^{n+1}_{Np}+y^n_{N+1,p}$$
$$\begin{aligned}\rho'(p_-)y^n_{Np} &= -\alpha_{Np}(-2\Lambda_1-2N-n+1)(-2\Lambda_1-2N+2-n)y^n_{N-1,p}\\ &\quad+\beta_{Np}n(-2\Lambda_1-2N-n+1)y^{n-1}_{Np}-n(n-1)y^{n-2}_{N+1,p}\end{aligned} \tag{5.4}$$
$$\begin{aligned}\rho'(k_-)y^n_{Np} &= -\delta_{Np}(-2\Lambda_1-2N-n+1)(-2\Lambda_1-2N-n+2)y^n_{N-1,p+1}\\ &\quad+n\gamma_{Np}(-2\Lambda_1-2N-n+1)y^{n-1}_{N,p+1}-n(n-1)y^{n-2}_{N+1,p}\end{aligned}$$
$$\rho'(k_+)y^n_{Np} = \delta_{Np}y^{n-2}_{N-1,p+1}+\gamma_{Np}y^{n+1}_{N,p+1}+y^n_{N+1,p+1}$$
$$\rho'(k_3)y^n_{Np} = -\delta_{Np}(-2\Lambda_1-2N-n+1)y^{n+1}_{N-1,p+1}-(-\Lambda_1-N-n)\gamma_{Np}y^n_{N,p+1}+ny^{n-1}_{N+1,p+1}$$
$$\rho'(k_0)y^n_{Np} = -(N-p)y^n_{N,p+1}$$

where

$$\alpha_{Np} = \frac{[\Lambda_2^2+(1-\Lambda_1-N)^2](N-p)(-2\Lambda_1+2-N-p)}{(-\Lambda_1+1-N)^2(-2\Lambda_1-2N+3)(-2\Lambda_1-2N+1)}$$

$$\beta_{Np} = \frac{-\Lambda_2(-\Lambda_1+1-p)}{(-\Lambda_1-N)(-\Lambda_1+1-N)}$$

$$\delta_{Np} = \frac{(N-p)(N-p-1)[\Lambda_2^2+(-\Lambda_1-N+1)^2]}{(-2\Lambda_1-2N+1)(-2\Lambda_1-2N+3)(-\Lambda_1-N+1)^2}$$

$$\gamma_{Np} = \frac{-(N-p)\Lambda_2}{(-\Lambda_1-N)(-\Lambda_1-N+1)}.$$

When restricted to the so(3, 1) subalgebra, one obtains the representations of so(3, 1) which were discussed in [2]. Further analysis of the above representations is analogous to the analysis given for the Lorentz algebra. Equations (5.4) also hold on the extended space Ω^e_+, $N, p, n \in \mathbb{Z}$.

Following the methods which were employed in § 3, one obtains the bases for finite dimensional quotient spaces for $\Lambda_1>0$, $-i\Lambda_2>0$, half-integers (case A):

$$\begin{aligned}&T_{(M/2,n/2,q_c)} = \{y^m_{Nq},\ q\leqslant N\leqslant \tfrac{1}{2}(n-M),\ -M-2N+1\leqslant m<0,\ 0\leqslant q\leqslant q_c\}\\ &M, n=3,5,7,\ldots,\ n>M, \quad 0\leqslant q_c\leqslant \tfrac{1}{2}(n-M)\end{aligned} \tag{5.5}$$

and for $\Lambda_1>0$, $-i\Lambda_2>0$ integers (case B):

$$\begin{aligned}&T_{(M,n,q_c)} = \{y^m_{Nq},\ q\leqslant N\leqslant n-M,\ -2M-2N+1\leqslant m<0,\ 0\leqslant q\leqslant q_c\}\\ &M, n=1,2,3,\ldots,\ n>M, \quad 0\leqslant q_c\leqslant n-M.\end{aligned} \tag{5.6}$$

In the following a few examples with lowest dimensionalities are shown:

$(M/2, n/2, q_c)=(\Lambda_1, -i\Lambda_2, q_c)$	(case A)
$(3/2, 5/2, 1)$	$10=6+4=(2+4)+4$
$(5/2, 7/2, 1)$	$16=10+6=(4+6)+6$
$(3/2, 7/2, 1)$	$22=12+10=(2+4+6)+(4+6)$
$(7/2, 9/2, 1)$	$22=14+8=(6+8)+8$
$(3/2, 7/2, 2)$	$28=12+10+6=(2+4+6)+(4+6)+6$
$(9/2, 11/2, 1)$	$28=18+10=(8+10)+10$
$(5/2, 9/2, 1)$	$32=18+14=(8+6+4)+(8+6)$

etc.

$(M, n, q_c)=(\Lambda_1, -i\Lambda_2, q_c)$	(case B)
$(1, 2, 1)$	$7=4+3=(1+3)+3$
$(2, 3, 1)$	$13=8+5=(5+3)+5$
$(1, 3, 1)$	$17=9+8=(1+3+5)+(3+5)$
$(3, 4, 1)$	$19=12+7=(5+7)+7$
$(1, 3, 2)$	$22=9+8+5=(1+3+5)+(3+5)+5$
$(4, 5, 1)$	$25=16+9=(9+7)+9$
$(2, 4, 1)$	$27=15+12=(3+5+7)+(5+7)$

etc.

A general formula for the dimensions of the finite dimensional indecomposable iso(3, 1) representations on Ω_- and Ω_+^e can be obtained. It is given as

$$d_{(\Lambda_1,\Lambda_2,q_c)}=\pm\tfrac{1}{6}(q_c+1)(q_c+2)(\pm 6\Lambda_1-4q_c+3)\pm(q_c+1)[(\pm\Lambda_1-q_c)^2+\Lambda_2^2)] \qquad (5.7)$$

where the upper sign corresponds to Ω_- and the lower sign to Ω_+^e.

One has to remember that the above holds only for those values of Λ_1, Λ_2, q_c that give finite dimensional indecomposable iso(3, 1) representations. For $q_c=0$ we get back the formula for the case of the Lorentz algebra.

6. Summary

Starting from the master representation for the Poincaré algebra, we obtained finite dimensional indecomposable iso(3, 1) representations on the translation subalgebra K with dimensions 5, 14, 15, ..., as well as on the 'lowering' algebra Ω_- and the 'raising' algebra Ω_+^e with dimensions 5, 7, 8, 10, 11, 13, 13′, 14, 14′, 16, 17, 17′, 18, 19, 20, 20′, 22, 22′, 22″, 23, 23′, 25, 25′, 26, 26′, 27, 28, 28′, 28″, 29, 29′, 30, ... The general formulae for the dimensions are given by equations (2.11) and (5.7), respectively. The so(3, 1) content was calculated for the representations with dimensions listed above. Moreover, the bases for the finite dimensional representations were obtained in an explicit form.

As far as infinite dimensional indecomposable iso(3, 1) representations are concerned, we confined ourselves to an example treated in appendix 2. The reader unfamiliar with the method we have used is referred to the simpler case of the Lorentz algebra treated in [1].

Appendix 1

In order to analyse the indecomposable representations in detail, one has to pay particular attention to those coefficients $\alpha_{...}$, $\beta_{...}$, $\gamma_{...}$, $\delta_{...}$, that vanish as well as to those that become singular. The same cases as in [1] will be discussed.

In what follows $\mathbb{N}$ stands for non-negative integers, $\mathbb{N}^+$ for positive integers and $\mathbb{N}^+_{\text{odd}}$ for positive odd integers.

Case A

$$\Lambda_1 = M/2, \qquad \Lambda_2 = \mathrm{i}n/2, \qquad M, n \in \mathbb{N}^+_{\text{odd}}.$$

(A1) $\Lambda_1 = 1/2, \qquad \Lambda_2 = \mathrm{i}/2\ (M = n = 1).$

The vanishing coefficients are α_{jj}, γ_{jj}, δ_{jj}, $\delta_{j+1,j}$, $j = 0, 1, 2, \ldots$, and α_{10}, α_{20}, α_{21} (see figure 2).

(A2) $\Lambda_1 = M/2, \qquad \Lambda_2 = \pm\mathrm{i}M/2, \qquad M \in \mathbb{N}^+_{\text{odd}} \quad \text{and} \quad M \geqslant 3.$

The vanishing coefficients are α_{jj}, γ_{jj}, δ_{jj}, $\delta_{j+1,j}$, $j = 0, 1, 2, \ldots$, and δ_{Nq}, α_{Nq} for $N = M+1$, $N = 1$, α_{Nq} for $N = M+2-q$. The singular bands mentioned in [1] are present here, namely for $N = (M+1)/2$, $N = (M+3)/2$ and q small. More precisely, singularities for $N = (M+1)/2$ occur if $q < (M+1)/2$ and singularities for $N = (M+3)/2$ occur if $q < (M+3)/2$.

(A3) $\Lambda_1 = M/2, \qquad \Lambda_2 = \pm\mathrm{i}/2 \qquad (n = 1, M \in \mathbb{N}^+_{\text{odd}}, M \geqslant 3).$

The vanishing coefficients are α_{jj}, γ_{jj}, δ_{jj}, $\delta_{j+1,j}$, $j = 0, 1, 2, \ldots$, and α_{Nq}, δ_{Nq} for $N = 1/2 + M/2 + 1$ and q arbitrary, α_{Nq} for $N = M+2-q$.

The singular bands appear for $N = (M+1)/2$ and $N = (M+3)/2$. Then the order of limits for the parameters involved becomes of importance (see [1]). In the following we confine our discussion to the case where the order of limits is taken in such a manner that $\alpha_{Nq} = \delta_{Nq} = 0$. This is always possible. For this case (as for case (A1)) the 'staircase' invariant subspaces of so(3, 1) occur as discussed in [1]. These are independent of the parameter q. For fixed value of q the elements of an invariant subspace of this kind are all y^m_{Nq} that lie above the 'staircase' defined by the equation:

$$2\Lambda_1 - 2N + 1 - m = 0$$

(see figure 3). It is the quotient spaces modulo these invariant subspaces which lead to *finite* dimensional indecomposable representations of iso(3, 1).

(A4) $\Lambda_1 = M/2, \qquad \Lambda_2 = \mathrm{i}n/2, \qquad M, n = 3, 5, 7, \ldots, \qquad n < M.$

The vanishing coefficients are α_{jj}, δ_{jj}, γ_{jj}, $\delta_{j+1,j}$, $j = 0, 1, 2, \ldots$, and α_{Nq}, δ_{Nq} for $N = \pm n/2 + M/2 + 1$, α_{Nq} for $N = M+2-q$. The singular bands occur for $N = (M+1)/2$, $N = (M+3)/2$.

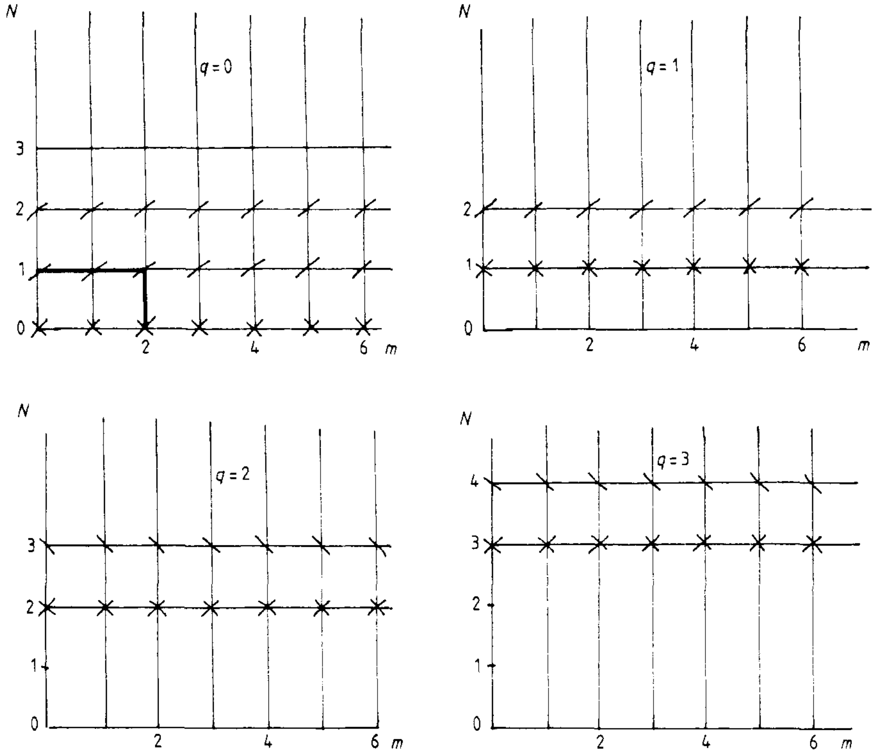

Figure 2. A sequence of invariant subspaces for the first few q ($q = 0, 1, 2, 3$) for case (A1). Description: ——— ($\alpha = 0$) moving down for fixed q forbidden, ///////// ($\alpha = 0$, $\delta = 0$) moving down for fixed q *and* q greater by 1 forbidden, \\\\\\\\\\ ($\delta = 0$) moving down for q greater by 1 forbidden, XXXXXXXXXXX ($\alpha = 0$, $\delta = 0$, $\gamma = 0$) moving down for fixed q forbidden *and* moving only up allowed for q greater by 1.

Case B

$$\Lambda_1 = M, \qquad \Lambda_2 = \pm i n, \qquad M, n \in \mathbb{N}.$$

$$\text{(B1)} \qquad \Lambda_1 = \Lambda_2 = 0 \qquad (M = n = 0).$$

The vanishing coefficients are α_{jj}, δ_{jj}, $\delta_{j+1,j}$, $j = 0, 2, 3, \ldots$ (except when $j = 1$), and α_{20}. The case when $j = 1$ takes care of the coefficients that *may* become singular, namely α_{11}, α_{10}, δ_{11}, δ_{10}, β_{00}, β_{10}, β_{11}, γ_{00}, γ_{10}, γ_{11}. Without explicitly listing the limits of the parameters involved we find the sets of self-consistent values as follows:

	α_{11}	α_{10}	δ_{11}	δ_{10}	β_{00}	β_{10}	β_{11}	γ_{00}	γ_{10}	γ_{11}
I	0	∞	0	0	∞	∞	0	0	∞	0
II	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
III	0	-1	0	0	0	0	0	0	0	0
IV	0	0	0	0	i	i	0	0	i	0

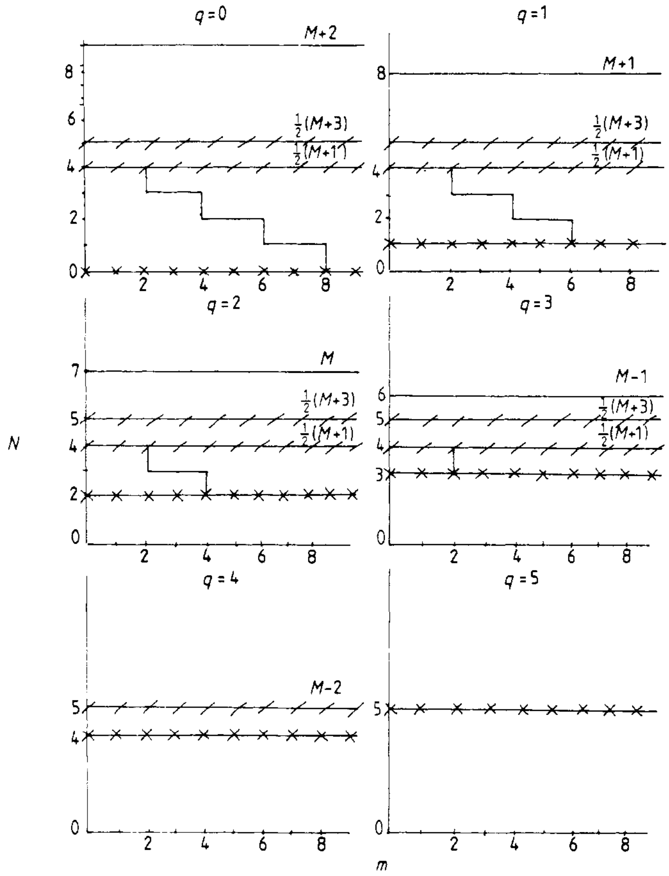

Figure 3. Case (A3) for $M = 7$ can be viewed as a representative picture for any M.

Cases III and IV yield distinct iso(3, 1) indecomposable representations and representations induced on quotient spaces can be extracted from them in the usual manner.

(B2) $\Lambda_1 = M, \qquad \Lambda_2 = \pm iM, \qquad M \in \mathbb{N}^+.$

The vanishing coefficients are α_{Nq} for $N = q$, $N = 2M+1$, $N = 1$, $N = 2M+2-q$, δ_{Nq} for $N = q$, $N = 2M+1$, $N = 1$, $N = q+1$, β_{Nq} for $q = M+1$, γ_{Nq} for $N = q$. Singularities occur for $N = M+1$ and $N = M$.

(B3) $\Lambda_1 = M, \qquad \Lambda_2 = \pm in, \qquad M \leqslant \mathbb{N}^+, \qquad n \leqslant M.$

The vanishing coefficients are α_{Nq} for $N = q$, $N = 2M+2-q$, $N = n+M+1$, δ_{Nq} for $N = q$, $N = q+1$, $N = n+M+1$, β_{Nq} for $q = M+1$, γ_{Nq} for $N = q$. Singularities occur for $N = M+1$, $N = M$.

Appendix 2

Using the result given in appendix 1 one can proceed with the analysis of the infinite dimensional indecomposable representations of iso(3, 1). Since in this paper we concentrate on finite dimensional indecomposable representations, we will only give an example of such analysis in case A1.

We introduce the following subspaces: $V^q = \{y^m_{q+N,q}, N, m \in \mathbb{N}\}$ where $q \in \mathbb{N}$. Each subspace V^q corresponds to the space Ω_- of so(3, 1) in [1], equation (4.18), and thus each subspace V^q carries so(3, 1) representations as was described in [1]. If the integer values of q are plotted along the z axis of a coordinate system, then the basis elements of each space V^q can be considered to lie in a plane which is parallel to the xy plane and has a corresponding integral coordinate q on the z axis. Furthermore, the integral values of m and $N+q$ will be represented by integral coordinates on the x axis and y axis, respectively. Then to each point $(m, q+N, q)$ of this integral coordinate system corresponds a basis vector of the space Ω_- of iso(3, 1) for $m, q+N, q \in \mathbb{N}$. It is noted that, as the value of q increases by 1, the minimal value of $q+N$ is shifted by 1 in the y direction.

From equation (3.4) it follows that the action of the operators ρ of iso(3, 1) on $y^m_{q+N,q}$, for $N=0$, maps these elements either on other elements of V^q (the action of the so(3, 1) subalgebra) or on elements $y^{m'}_{q+1+N,q+1}$ of V^{q+1}. The latter is true since $\delta_{qq} = \gamma_{qq} = 0$ and since $\rho(k_0)$ maps these elements to zero. Moreover, since $\delta_{q+1,q} = 0$, the elements $y^m_{q+1+N,q}$, for $N=0$, are either mapped onto elements of the space V^q (the so(3, 1) subalgebra) or onto linear combinations of elements $y^m_{q+1+N,q+1}$ and $y^{m'}_{q+2+N,q+1}$ (again $N=0$) of V^{q+1}. It is then easy to realise that the space

$$V = \bigcup_q V^q \qquad (q \in \mathbb{N})$$

forms a space which is invariant (but not irreducible) with respect to the action ρ of the representation equation (3.4). In fact, equation (4.2) gives a decomposition of Ω_- with respect to so(3, 1) invariant subspaces. Each of these so(3, 1) invariant subspaces exhibits the properties of the so(3, 1) representations discussed in [1] for the appropriate choice of parameters Λ_1, Λ_2.

In addition to the subspaces V^q of Ω_- we introduce the subspaces

$$V^q_{q+R} = \{y^m_{q+r,q}, r \geqslant R, m \in \mathbb{N}\}$$

where $q, R \in \mathbb{N}$. The spaces V^q_{q+R} form subspaces of the spaces V^q. They are obtained from the spaces V^q by deleting the basis elements $y^m_{q+N,q}$ for $N = 0, 1, \ldots, R-1$. In particular, $V^q_q \equiv V^q$. These subspaces will be needed in what follows.

Below we will list iso(3, 1) invariant subspaces of Ω_-:

$$\Omega_- = V^0_0 \cup V^1_1 \cup V^2_2 \cup \ldots = \bigcup_q V^q_q$$

$$W_1 = V^0_1 \cup V^1_1 \cup V^2_2 \ldots$$

$$W_2 = V^0_2 \cup V^1_1 \cup V^2_2 \ldots$$

$$W_3 = V^0_2 \cup V^1_2 \cup V^2_2 \ldots$$

$$W_4 = V^0_1 \cup V^1_2 \cup V^2_2 \ldots$$

$$W_5 = V^0_3 \cup V^1_1 \cup V^2_2 \ldots$$

$$W_6 = V^0_3 \cup V^1_2 \cup V^2_2 \ldots$$

$$W_7 = V^1_1 \cup V^2_2 \cup V^3_3 \ldots$$

$$W_8 = V_2^1 \cup V_2^2 \cup V_3^3 \ldots$$

$$\vdots$$

There are infinitely many of them. Making use of those invariant subspaces one can construct infinitely many quotient spaces. These quotient spaces carry infinite dimensional indecomposable iso(3, 1) representations. For example (the slash \ denotes the difference of sets),

$$T_0^1 = \Omega_-/(\Omega_- \setminus V_0^0) \cong V_0^0$$

$$T_0^2 = \Omega_-/(\Omega_- \setminus V_0^0 \setminus V_1^1) \cong V_0^0 \cup V_1^1$$

$$T_0^3 = \Omega_-/(\Omega_- \setminus V_0^0 \setminus V_1^1 \setminus V_2^2) \cong V_0^0 \cup V_1^1 \cup V_2^2$$

etc,

$$T_1^1 = W_1/(W_1 \setminus V_1^0) \cong V_1^0$$

$$T_1^2 = W_1/(W_1 \setminus V_1^0 \setminus V_1^1) \cong V_1^0 \cup V_1^1$$

etc.

These quotient spaces have, in turn, invariant subspaces. We choose T_0^2 as an example. Its invariant subspaces are $L_1 = V_1^0 \cup V_1^1$, $L_2 = V_2^0 \cup V_1^1$, $L_3 = V_3^0 \cup V_1^1$, $L_4 = V_1^0 \cup V_2^1$, $L_5 = V_2^0 \cup V_2^1$, $L_6 = V_3^0 \cup V_2^1$, $L_7 = V_1^1$, $L_8 = V_2^1$. In addition there exist the invariant subspaces (due to the existence of the non-trivial $\rho(h_+)$ extremal vector y_3^1), $L_9 = L \cup L_1$, $L_{10} = L \cup L_2$, $L_{11} = L \cup L_3$, $L_{12} = L \cup L_4$, $L_{13} = L \cup L_7$, where $L = \{y_{00}^{2+m}, m \in \mathbb{N}\}$. The quotient spaces of T_0^2 are then

(1) $T_0^2/(V_1^0 \cup V_1^1) \cong \{y_{00}^m, m \in \mathbb{N}\}$ carrying an infinite dimensional indecomposable so(3) representation with highest weight 1/2,

(2) $T_0^2/(V_2^0 \cup V_2^1) \cong \{y_{00}^m, y_{10}^m, y_{11}^m, m \in \mathbb{N}\}$ carrying an infinite dimensional indecomposable iso(3, 1) representation with its so(3) content consisting of infinite dimensional irreducible representations with highest weights $-1/2$, $-1/2$ and the so(3) indecomposable representation with highest weight 1/2,

(3) $T_0^2/(V_3^0 \cup V_2^1) = \{y_{00}^m, y_{10}^m, y_{20}^m, y_{11}^m, m \in \mathbb{N}\}$ carrying an infinite dimensional indecomposable iso(3, 1) representation with its so(3) content consisting of infinite dimensional irreducible representations with highest weights $-1/2$, $-1/2$, $-3/2$ and the so(3) indecomposable representation with highest weight 1/2.

The analysis of other quotient spaces is similar. One obtains infinite dimensional irreducible and indecomposable iso(3, 1) representations, whose so(3) content can be determined in a straightforward manner. We confine ourselves to a listing of all remaining quotient spaces of T_0^2:

(4) $T_0^2/(V_2^0 \cup V_1^1) \cong \{y_{00}^m, y_{10}^m, m \in \mathbb{N}\}$
(5) $T_0^2/(V_3^0 \cup V_1^1) \cong \{y_{00}^m, y_{10}^m, y_{20}^m, m \in \mathbb{N}\}$
(6) $T_0^2/V_1^1 \cong V_0^0$
(7) $T_0^2/V_2^1 \cong V_0^0 \cup \{y_{11}^m, m \in \mathbb{N}\}$
(8) $T_0^2/(V_1^0 \cup V_2^1) \cong \{y_{00}^m, y_{11}^m, m \in \mathbb{N}\}$
(9) $T_0^2/L_9 \cong \{y_{00}, y_{00}^1\}$
(10) $T_0^2/L_{10} \cong \{y_{00}, y_{00}^1\} \cup \{y_{10}^m, m \in \mathbb{N}\}$
(11) $T_0^2/L_{11} \cong \{y_{00}, y_{00}^1\} \cup \{y_{10}^m, y_{20}^m, m \in \mathbb{N}\}$
(12) $T_0^2/L_{12} \cong \{y_{00}, y_{00}^1\} \cup \{y_{11}^m, m \in \mathbb{N}\}$
(13) $T_0^2/L_{13} \cong \{y_{00}, y_{00}^1\} \cup V_1^0$.

The representation induced for case (9) is finite dimensional, but trivial ($\rho(k) = 0$).

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